Game Theory - Intro

- What is Game Theory?

  A branch of mathematics (decision theory), which formalizes games and defines solutions to them
- What is a Game?
- It is a decision problem, where decision-maker's payoff (profit) **may depend** not only on his own decision, but also on the decisions made by other decision makers

Defining a game

- **Formally,** a game **is a set of 4 elements:**
  - a set of players (can even be infinite)
  - a set of rules (allowable actions and sequencing of actions by each player)
  - a payoff function (which assigns payoffs for each player as a function of strategies chosen)
  - informational structure (what players know at each point in the game)

General Assumptions

- Standard GT **assumes** that players are:
  - **selfish**: maximize their own payoffs and do not care about the opponent's payoffs
  - **rational**: they understand the game and can determine the optimal strategy
  - **expected-utility maximizers**: in uncertain situations players they base their choices on (von Neumann-Morgenstern) expected utility
  - share **common knowledge** about all aspects of the game
  - in addition, it is often assumed that players **do not communicate, cooperate or negotiate**, unless the game allows it explicitly

More on Assumptions

- All of the above are simplifying assumptions, i.e. they rarely hold in reality
- There is a lot of research on games with altruistic players, players with bounded-rationality or non-expected-utility maximizers or even non-decision makers (e.g. *Evolutionary Game Theory)*
- A whole separate branch of decision theory deals with cooperative games (*Cooperative Game Theory*)

More on Common Knowledge

- "As we know, there are **known knowns.** These are things we know we know.
- We also know, there are **known unknowns.** That is to say we know there are some things we do not know.
- But there are also **unknown unknowns,** the ones we don't know we don't know".
- **D.H. Rumsfeld,** *Feb. 12, 2002, Department of Defense news briefing*
- **Common knowledge** means that there are **no unknown unknowns**.

Incomplete Information vs Asymmetric Knowledge

- Modeling asymmetric knowledge (unknown unknowns) is difficult
- Instead, Game theorists assume that if a player doesn't know something, she has some initial beliefs about it and these beliefs are commonly known (there are only known unknowns)
- Games with known unknowns are called **games with incomplete (imperfect) information**.

History of Game Theory

- Cournot (1838) - quantity-setting duopoly model
- Bertrand (1883) – price-setting duopoly model
- Zermelo (1913) – the game of chess
- von Neumann & Morgenstern (1944) – defined games, min-max solution for 0-sum games
- Nash (1950) – defined a the equilibrium and the solution to a cooperative bargaining problem

‘Nobel’ prize winners for Game Theory (Economics)

- 1994 – John Nash, John Harsanyi, Reinhard Selten
- 1996 – James Mirrlees, William Vickrey
- 2005 – Robert Aumann, Thomas Shelling
- 2007 – Leonid Hurwicz, Eric Maskin, Roger Myerson,

Normal Form Games

- Simple games without „timing”, i.e. where players make decisions simultaneously. Dynamic games can be reduced into a normal form.
- The set of strategies is simply the set of possible choices for each player.
- Normal (Strategic) Form Game consists of the following elements:
- - $N=\{1,.., n\}$ the finite set of players
- - $S=\{S_1,.., S_n\}$ the set of strategies, including a (possibly infinite) set for each player
- - $U(s_1,.., s_n)$ the vector of payoff functions, where $s_i \in S_i$ for each player
- If the set of strategies is small and countable (typically 2-5), then we can use a **game matrix** to represent a normal-form game

Game Matrix

- Example 1: Advertising game

		Player 2	
		A	N
Player 1	A	40, 40	60, 30
	N	30, 60	50, 50

- N={1, 2}
- S={S_1, S_2} and $S_1 = S_2 = \{A, N\}$
- $U(s_1, s_2) = \{ u_1(s_1, s_2); u_2(s_1, s_2)\}$
- $u_1(A, A) = 40; u_1(A, N) = 60; u_1(N, A) = 30; u_1(N, N) = 50$
- $u_2(A, A) = 40; u_2(A, N) = 30; u_2(N, A) = 60; u_2(N, N) = 50$

Mixed Strategies

- In any game, but especially in games such as above (with countable strategies), it is often useful to consider **mixed strategies**
- Mixed strategies are a probability distribution over the set of (pure) strategies S, a convex extension of that set.
- The set of mixed strategies is denoted by $\sum=\{\sum_1, \sum_2\}$, a single strategy is of player *i* is denoted by $\sigma_i \in \sum_i$
- We simply allow the players to make a random choice, with any possible probability distribution over the set of choices.

Dominance

- σ_{-i} = vector of mixed strategies of players other than *i*
- Def:
  Pure strategy s_i is **strictly** dominated (never-best-response), if for every σ_{-i} there is a strategy $z_i \in \sum_i$ of player i s.t. $u_i(z_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$
- There is also a notion of **weak dominance**, where it is enough that the strategy z_i is never worse (but doesn't have to be always better) than s_i
- **Iterated elimination of dominated strategies (IEDS)** is a simple procedure that provides a solution to many normal-form games
  - Step 1: Identify all dominated strategies
  - Step 2: Eliminate them to obtain a reduced game
  - Step 3: Go to Step 1

Iterated Elimination of Dominated Strategies

- In the Advertising game, the IEDS solution is (A, A)
- What about the game below?

	Player 2		
		L	*R*
	U	3, *1*	0, *2*
Player 1	M	0, *0*	3, *1*
	D	1, *2*	1, *1*

- D is dominated by a mixed strategy (e.g. 50-50 mix od U-M), then L is dominated by R, then U by M, solution: M-R